COVARIANT VERSION OF THE STINESPRING TYPE THEOREM FOR HILBERT C^*-MODULES

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Abstract. We prove a covariant version of the Stinespring theorem for Hilbert C^*-modules.

1. Introduction

A completely positive linear map from a C^*-algebra A to another C^*-algebra B is a map $\varphi : A \to B$ with the property that $[\varphi(a_{ij})]_{i.j=1}^n$ is a positive element in the C^*-algebra $M_n(B)$ of all $n \times n$ matrices with elements in B for all positive matrices $[a_{ij}]_{i.j=1}^n$ in $M_n(A)$ and for all positive integers n. The study of completely positive maps is motivated by the applications of the theory of completely positive maps to quantum information theory (operator valued completely positive maps on C^*-algebras are used as mathematical model for quantum operations) and quantum probability.

Sitinespring [9] shown that a completely positive map $\varphi : A \to L(H)$ is of the form $\varphi(\cdot) = V^*\pi(\cdot)V$ where π is a $*$-representation of A on a Hilbert space K and V is a bounded linear operator from H to K.

Hilbert C^*-modules are generalizations of Hilbert spaces and C^*-algebras. In [3] it is proved a version of the Stinespring theorem for completely positive map on Hilbert C^*-modules. In this paper, we will prove a version of the covariant Stinespring theorem for Hilbert C^*-modules.

A Hilbert C^*-module X over a C^*-algebra A (or a Hilbert A-module) is a linear space that is also a right A-module, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle$ that is $\mathbb{C}$- and A-linear in the second variable and conjugate linear in the first variable such that X is complete with the norm $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$. X is full if the closed bilateral $*$-sided ideal $\langle X,X\rangle$ of A generated by $\{\langle x,y\rangle ; x,y \in X\}$ coincides with A.

A representation of X on the Hilbert spaces H and K is a map $\pi_X : X \to L(H,K)$ with the property that there is a $*$-representation π_A of A on the Hilbert space H such that

$$\langle\pi_X(x),\pi_X(y)\rangle = \pi_A(\langle x,y\rangle)$$

for all $x,y \in X$. If X is full, then the $*$-representation π_A associated to π_X is unique. A representation $\pi_X : X \to L(H,K)$ of X is *nondegenerate* if $[\pi_X(X)H] = K$ and $[\pi_X(X)^*K] = H$ (here, $[Y]$ denotes the closed subspace of a Hilbert space Z generated by the subset $Y \subseteq Z$). Two representations $\pi_X : X \to L(H,K)$ and

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$\pi_X^{'} : X \to L(H^{'}, K^{'})$ are *unitarily equivalent* if there are two unitary operators $U_1 \in L(H, H^{'})$ and $U_2 \in L(K, K^{'})$ such that $U_2 \pi_X(x) = \pi_X^{'}(x) U_1$ for all x in X [1].

A map $\Phi : X \to L(H, K)$ is called *a completely positive map on* X if there is a completely positive linear map $\varphi : A \to L(H)$ such that

$$\langle \Phi\left(x\right), \Phi\left(y\right) \rangle = \varphi\left(\langle x, y \rangle\right)$$

for all x and y in X. If X is full, then the completely positive map φ associated to Φ is unique. If $\Phi : X \to L(H, K)$ is a completely positive map on X, then Φ is linear and continuous.

B.V.R. Bhat, G. Ramesh, and K. Sumesh [3] provided a Stinespring construction associated to a completely positive map Φ on a Hilbert C^*-module X in terms of the Stinespring construction associated to the underlying completely positive map φ. In Section 2 we present a Stinespring construction associated to a completely positive map on a full Hilbert C^*-module, the construction is similar to the construction given in [3, Theorem 2.1] but it is not given in terms of the underlying completely positive map.

A *morphism of Hilbert C^*-modules* [2] or a generalized isometry [10] is a map $\Psi : X \to Y$ from a Hilbert A-module X to a Hilbert B-module Y with the property that there is a C^*-morphism $\psi : A \to B$ such that

$$\langle \Psi\left(x\right), \Psi\left(y\right) \rangle = \psi\left(\langle x, y \rangle\right)$$

for all x and y in X. If X is full, then the underlying C^*-morphism of Ψ is unique, in fact Ψ is a ternary morphism [10, Theorem 2.1]. A map $\Psi : X \to Y$ is *an isomorphism of Hilbert C^*-modules* if it is invertible, Ψ and Ψ^{-1} are morphisms of Hilbert C^*-modules.

Suppose that G is a locally compact group, Δ is the modular function of G with respect to left invariant Haar measure ds. *A continuous action* of G on a full Hilbert A-module X is a group morphism $t \mapsto \eta_t$ from G to $\mathrm{Aut}(X)$, the group of all isomorphisms of Hilbert C^*-modules from X to X, such that the map $(t, x) \mapsto \eta_t\left(x\right)$ from $G \times X$ to X is continuous. The triple (G, η, X) will be called a dynamical system on Hilbert C^*-modules. Any C^*-dynamical system (G, α, A) can be regarded as a dynamical system on Hilbert C^*-modules.

Let $t \mapsto u_t$ and $t \mapsto u_t'$ be two unitary $*$-representations of G on the Hilbert spaces H and K. A completely positive map $\Phi : X \to L(H, K)$ is (u', u)-*covariant* with respect to (G, η, X) if

$$\Phi\left(\eta_t\left(x\right)\right) = u_t' \Phi\left(x\right) u_t^*$$

for all $x \in X$ and for all $t \in G$. Clearly, if $\Phi : A \to L(H)$ is a completely positive map u-covariant with respect to the C^*-dynamical system (G, α, A), then it is (u, u)-covariant with respect to the dynamical system on Hilbert C^*-modules, (G, α, A).

In Section 3, we provide a covariant version of the Stinespring theorem, and in Section 4, we show that any covariant completely positive map Φ with respect to (G, η, X) induces a completely positive map on the crossed product $G \times_\eta X$.

2. The Stinespring type theorem for Hilbert C^*-modules

Proposition 2.1. *Let $\pi_X : X \to L(H,K)$ be a representation of X, $V \in L(H)$ and $W \in L(K)$ a coisometry. Then the map $\Phi : X \to L(H,K)$ defined by*

$$\Phi(x) = W^* \pi_X(x) V$$

for all $x \in X$ is a completely positive map.

Proof. Indeed, we have

$$\begin{aligned} \langle \Phi(x), \Phi(y) \rangle &= \langle W^* \pi_X(x) V, W^* \pi_X(y) V \rangle \\ &= \langle \pi_X(x) V, \pi_X(y) V \rangle = V^* \pi_A(\langle x, y \rangle) V \end{aligned}$$

for all $x, y \in X$, and since the map $\varphi : A \to L(H)$ defined by

$$\varphi(a) = V^* \pi_A(a) V$$

is completely positive, Φ is completely positive. □

We show that an operator valued completely positive linear map Φ on a full Hilbert C^*-module X is of the form $\Phi(\cdot) = W^* \pi_X(\cdot) V$, where π_X is a representation of X, W is a coisometry and V is a bounded linear map. Moreover, under some conditions this writing is unique up to unitary equivalence.

Theorem 2.2. *Let X be a full Hilbert C^*-module over a C^*-algebra A, H and K two Hilbert spaces and $\Phi : X \to L(H,K)$ a completely positive map. Then:*

(1) *There are two Hilbert spaces H_Φ and K_Φ, a representation $\pi_\Phi : X \to L(H_\Phi, K_\Phi)$ of X, a bounded linear operator $V_\Phi : H \to H_\Phi$ and a coisometry $W_\Phi : K \to K_\Phi$ such that:*
   (a) $\Phi(x) = W_\Phi^* \, \pi_\Phi(x) V_\Phi$ *for all $x \in X$;*
   (b) $[\pi_\Phi(X) V_\Phi H] = K_\Phi$;
   (c) $\left[\pi_\Phi(X)^* W_\Phi K\right] = H_\Phi$.

(2) *If H' and K' are two Hilbert spaces, $\pi_X : X \to L(H', K')$ a representation of X, V' an element in $L(H, H')$ and $W' : K \to K'$ a coisometry that verify the following relations:*
   (a) $\Phi(x) = W'^* \, \pi_X(x) V'$ *for all $x \in X$;*
   (b) $[\pi_X(X) V' H] = K'$;
   (c) $\left[\pi_X(X)^* W' K\right] = H'$,

then there are two unitary operators $U_1 \in L(H_\Phi, H')$ and $U_2 \in L(K_\Phi, K')$ such that: $U_2 \pi_\Phi(x) = \pi_X(x) U_1$ for all $x \in X$, $V' = U_1 V_\Phi$ and $W' = U_2 W_\Phi$.

Proof. (1) Let φ be the completely positive linear map associated to Φ and let $(\pi_\varphi, H_\varphi, V_\varphi)$ be the Stinespring construction associated to φ [7, Theorem 5.6 (1)]. Let $H_\Phi = H_\varphi$, $V_\Phi = V_\varphi, K_\Phi = [\Phi(X)H]$ and W_Φ the projection of K on K_Φ. Exactly as in the proof of Theorem 2.1 [3] it is shown that the map $\pi_\Phi : X \to L(H_\Phi, K_\Phi)$ defined by $\pi_\Phi(x)\left(\sum\limits_{i=1}^{n} \pi_\varphi(a_i) V_\Phi h_i\right) = \sum\limits_{i=1}^{n} \Phi(x a_i) h_i$ is a representation of X that verifies the relations (a) and (b). From

$$\begin{aligned} \left[\pi_\Phi(X)^* W_\Phi K\right] &= \left[\pi_\Phi(X)^* K_\Phi\right] = \left[\pi_\Phi(X)^* \pi_\Phi(X) V_\Phi H\right] \\ &= \left[\pi_\varphi(\langle X, X \rangle) V_\Phi H\right] = \left[\pi_\varphi(A) V_\Phi H\right] = H_\Phi \end{aligned}$$

we deduce that the relation (c) is verified too.

(2) If $\pi_A:A \to L(H')$ is the $*$-representation associated to π_X, then:

$$\begin{aligned}\varphi\left(\langle x,y\rangle\right) &= \langle \Phi\left(x\right),\Phi\left(y\right)\rangle = \left(W'^{*}\pi_X\left(x\right)V'\right)^{*}W'^{*}\pi_X\left(y\right)V' \\ &= V'^{*}\pi_X\left(x\right)W'W'^{*}\pi_X\left(y\right)V' = V'^{*}\pi_A\left(\langle x,y\rangle\right)V'\end{aligned}$$

for all x and y in X, and

$$\begin{aligned}\left[\pi_A\left(\langle X,X\rangle\right)V'H\right] &= \left[\pi_X\left(X\right)^{*}\pi_X\left(X\right)V'H\right] = \left[\pi_X\left(X\right)^{*}K'\right] \\ &= \left[\pi_X\left(X\right)^{*}W'K\right] = H'.\end{aligned}$$

Therefore, (π_A, H', V') is unitarily equivalent to the Stinespring construction associated to φ [7, Theorem 5.6 (2)], and so there is a unitary operator $U_1 \in L(H_\Phi, H')$ such that $\pi_A(a) = U_1\pi_\varphi\left(a\right)U_1^{*}$ and $V' = U_1V_\Phi$. As in the proof of Theorem 2.4 [3] we show that the there is a unitary operator $U_2 : K_\Phi \to K'$ such that

$$U_2\left(\sum_{i=1}^{n}\pi_\Phi\left(x_i\right)V_\Phi h_i\right) = \sum_{i=1}^{n}\pi_X\left(x_i\right)V'h_i$$

and moreover,

$$U_2\pi_\Phi\left(x\right) = \pi_X\left(x\right)U_1 \text{ and } W' = U_2W_\Phi.$$

□

3. The covariant version of the Stinespring construction

Let (G, η, X) be a dynamical system on Hilbert C^*-modules. *A covariant representation* of (G, η, X) is a quadruple (π_X, v, w, H, K) consists of two Hilbert spaces H and K, a representation $\pi_X : X \to L(H, K)$ of X, a unitary $*$-representation of G on H, $t \mapsto v_t$, and a unitary $*$-representation of G on K, $t \mapsto w_t$ such that

$$\pi_X\left(\eta_t\left(x\right)\right) = w_t\pi_X\left(x\right)v_t^{*}$$

for all $x \in X$ and for all $t \in G$. We say that the covariant representation (π_X, v, w, H, K) is nondegenerate if the representation π_X is nondegenerate. Clearly, any covariant representation of a C^*-dynamical system (G, α, A) is a covariant representation of (G, α, A) regarded as dynamical system on Hilbert C^*-modules.

Any continuous action $t \mapsto \eta_t$ of G on X induces a unique continuous action $t \mapsto \alpha_t^\eta$ of G on A such that $\alpha_t^\eta\left(\langle x,y\rangle\right) = \langle \eta_t\left(x\right), \eta_t\left(x\right)\rangle$ for all $x, y \in X$ and for all $t \in G$ [4].

Remark 3.1. *A (nondegenerate) covariant representation (π_X, v, w, H, K) of (G, η, X) induces a (nondegenerate) representation of (G, α^η, A). Indeed, if π_A is the $*$-representation associated to π_X, then*

$$\begin{aligned}\pi_A\left(\alpha_t^\eta\left(\langle x,y\rangle\right)\right) &= \pi_A\left(\langle \eta_t\left(x\right), \eta_t\left(y\right)\rangle\right) = \langle \pi_X\left(\eta_t\left(x\right)\right), \pi_X\left(\eta_t\left(y\right)\right)\rangle \\ &= \langle w_t\pi_X\left(x\right)v_t^{*}, w_t\pi_X\left(y\right)v_t^{*}\rangle = v_t\pi_A\left(\langle x,y\rangle\right)v_t^{*}\end{aligned}$$

for all $x, y \in X$ and for all $t \in G$. Therefore (π_A, v, H) is a covariant representation of (G, α^η, A).

Let $t \mapsto u_t$ and $t \mapsto u_t'$ be two unitary $*$-representations of G on the Hilbert spaces H and K.

Remark 3.2. *If $\Phi : X \to L(H, K)$ is a completely positive map, (u', u)-covariant with respect to (G, η, X), then the completely positive map φ associated to Φ is u-covariant with respect to (G, α^η, A).*

Indeed, we have

$$\begin{aligned}\varphi\left(\alpha_t^\eta\left(\langle x,y\rangle\right)\right) &= \varphi\left(\left\langle \eta_t\left(x\right),\eta_t\left(y\right)\right\rangle\right)=\left\langle \Phi\left(\eta_t\left(x\right)\right),\Phi\left(\eta_t\left(y\right)\right)\right\rangle \\ &= \left\langle u_t'\Phi\left(x\right)u_t^*,u_t'\Phi\left(y\right)u_t^*\right\rangle = u_t\varphi\left(\langle x,y\rangle\right)u_t^*\end{aligned}$$

for all $x, y \in X$ and for all $t \in G$.

Proposition 3.3. *Let (π_X, v, w, H, K) be a covariant representation of (G, η, X), $V \in L(H)$, $W \in L(K)$ a coisometry, $t \mapsto u_t$ and $t \mapsto u_t'$ two unitary $*$-representations of G on H respectively K such that $v_tV = Vu_t$ and $w_tW = Wu_t'$ for all $t \in G$. Then the map $\Phi : X \to L(H, K)$ defined by*

$$\Phi\left(x\right) = W^*\pi_X\left(x\right)V$$

for all $x \in X$ is a completely positive map, (u', u)-covariant with respect to (G, η, X).

Proof. By Proposition 2.1, the map Φ is completely positive. From

$$\Phi\left(\eta_t\left(x\right)\right) = W^*\pi_X\left(\eta_t\left(x\right)\right)V = W^*w_t\pi_X\left(x\right)v_t^*V = u_t'W^*\pi_X\left(x\right)Vu_t^* = u_t'\Phi\left(x\right)u_t^*$$

for all $x \in X$ and for all $t \in G$, we deduce that the completely positive map Φ is (u', u)-covariant. □

We show that an operator valued (u', u)-covariant completely positive map Φ on a full Hilbert C^*-module X is of the form $\Phi(\cdot) = W^*\pi_X(\cdot)V$, where (π_X, v, w, H, K) is a covariant representation of (G, η, X), W is a coisometry such that $w_tW = Wu_t'$ for all $t \in G$ and V is a bounded linear map such that $v_tV = Vu_t$ for all $t \in G$. Moreover, under some conditions this writing is unique up to unitary equivalence.

Theorem 3.4. *Let $\Phi : X \to L(H, K)$ be a completely positive map, (u', u)-covariant with respect to (G, η, X). Then:*

(1) *There are two Hilbert spaces H_Φ and K_Φ, a covariant representation $(\pi_\Phi, v^\Phi, w^\Phi, H_\Phi, K_\Phi)$ of (G, η, X), a linear operator $V_\Phi : H \to H_\Phi$ and a coisometry $W_\Phi : K \to K_\Phi$ such that:*
 (a) $\Phi\left(x\right) = W_\Phi^*\ \pi_\Phi\left(x\right)V_\Phi$ *for all $x \in X$;*
 (b) $v_t^\Phi V_\Phi = V_\Phi u_t$ *for all $t \in G$;*
 (c) $w_t^\Phi W_\Phi = W_\Phi u_t'$ *for all $t \in G$*
 (d) $\left[\pi_\Phi\left(X\right)V_\Phi H\right] = K_\Phi$;
 (e) $\left[\pi_\Phi\left(X\right)^* W_\Phi K\right] = H_\Phi$.

(2) *If H' and K' are two Hilbert spaces, (π_X, v, w, H', K') a covariant representation of (G, η, X), V' an element in $L(H, H')$ and $W' : K \to K'$ a coisometry which verify the following relations:*
 (a) $\Phi\left(x\right) = W'^*\ \pi_X\left(x\right)V'$ *for all $x \in X$;*
 (b) $v_tV' = V'u_t$ *for all $t \in G$;*
 (c) $w_tW' = W'u_t'$ *for all $t \in G$;*
 (d) $\left[\pi_X\left(X\right)V'H\right] = K'$;
 (e) $\left[\pi_X\left(X\right)^* W'K\right] = H'$,
 then there are two unitary operators $U_1 \in L(H_\Phi, H')$ and $U_2 \in L(K_\Phi, K')$ such that: $U_2\pi_\Phi\left(x\right) = \pi_X\left(x\right)U_1$, $v_tU_1 = U_1v_t^\Phi$, $w_tU_2 = U_2w_t^\Phi$, $V_t' = U_1V_\Phi$ and $W' = U_2W_\Phi$.

Proof. (1) Let φ be the completely positive map associated to Φ. Then, by Remark 3.2, φ is u-covariant with respect to (G, α^η, A). Let $(\pi_\varphi, v^\varphi, H_\varphi, V_\varphi)$ be the covariant Stinespring construction associated to φ (see, for example, [8]). Then

$(\pi_\varphi, H_\varphi, V_\varphi)$ is the Stinespring construction associated to φ and by Theorem 2.2, $(\pi_\Phi, H_\Phi, K_\Phi, V_\Phi, W_\Phi)$, where $H_\Phi = H_\varphi$, $V_\Phi = V_\varphi$, $K_\Phi = [\Phi(X)H]$, W_Φ is the projection of K on K_Φ and $\pi_\Phi : X \to L(H_\Phi, K_\Phi)$ is defined by $\pi_\Phi\left(x\right)\left(\sum\limits_{i=1}^{n}\pi_\varphi\left(a_i\right)V_\Phi h_i\right)$ $= \sum\limits_{i=1}^{n}\Phi\left(xa_i\right)h_i$ is the Sinespring construction associated to Φ. Moreover, the relations (a), (d) and (e) are verified.

Let $v_t^\Phi = v_t^\varphi$ for all $t \in G$. Then $t \mapsto v_t^\Phi$ is a unitary $*$-representation of G on H_Φ which verifies the relation (b). Since Φ is (u', u)-covariant, $u'_t\left(\sum\limits_{i=1}^{n}\Phi\left(x_i\right)h_i\right)$ $= \sum\limits_{i=1}^{n}\Phi\left(\eta_t\left(x_i\right)\right)u_t h_i$ for all $t \in G$ and for all $x_i \in X, h_i \in H$, $i = 1, ..., n$, and so $[\Phi(X)H]$ is invariant under u'. Then, since W_Φ is the projection on $[\Phi(X)H]$, we have $u'_t W_\Phi = W_\Phi u'_t$. Let $w_t^\Phi = u'_t|_{K_\Phi}$ for all $t \in G$. Then $t \mapsto w_t^\Phi$ is a unitary $*$-representation of G on K_Φ which verifies the relation (c).

To prove the assertion (1) it remains to show that $\left(\pi_\Phi, v^\Phi, w^\Phi, H_\Phi, K_\Phi\right)$ is a covariant representation of (G, η, X). From

$$\begin{aligned}\pi_\Phi\left(\eta_t\left(x\right)\right)\left(\sum_{i=1}^{n}\pi_\varphi\left(a_i\right)V_\varphi h_i\right) &= \sum_{i=1}^{n}\Phi\left(\eta_t\left(x\right)a_i\right)h_i = \sum_{i=1}^{n}\Phi\left(\eta_t\left(x\alpha_{t^{-1}}^{\eta}\left(a_i\right)\right)\right)h_i \\ &= \sum_{i=1}^{n}u'_t\Phi\left(x\alpha_{t^{-1}}^{\eta}\left(a_i\right)\right)u_t^* h_i\end{aligned}$$

and

$$\begin{aligned}w_t^\Phi\pi_X\left(x\right)v_{t^{-1}}^\Phi\left(\sum_{i=1}^{n}\pi_\varphi\left(a_i\right)V_\varphi h_i\right) &= w_t^\Phi\pi_X\left(x\right)\left(\sum_{i=1}^{n}v_{t^{-1}}^\Phi\pi_\varphi\left(a_i\right)V_\varphi h_i\right) \\ &= w_t^\Phi\pi_X\left(x\right)\left(\sum_{i=1}^{n}\pi_\varphi\left(\alpha_{t^{-1}}^{\eta}\left(a_i\right)\right)v_{t^{-1}}^\Phi V_\varphi h_i\right) \\ &= w_t^\Phi\pi_X\left(x\right)\left(\sum_{i=1}^{n}\pi_\varphi\left(\alpha_{t^{-1}}^{\eta}\left(a_i\right)\right)V_\varphi u_t^* h_i\right) \\ &= \sum_{i=1}^{n}u'_t\Phi\left(x\alpha_{t^{-1}}^{\eta}\left(a_i\right)\right)u_t^* h_i\end{aligned}$$

for all $a_1, ..., a_n \in A$ and for all $h_1, ..., h_n \in H$, we deduce that $\pi_\Phi\left(\eta_t\left(x\right)\right) = w_t^\Phi\pi_X\left(x\right)v_{t^{-1}}^\Phi$ for all $x \in X$ and for all $t \in G$. Therefore, $\left(\pi_\Phi, v^\Phi, w^\Phi, H_\Phi, K_\Phi\right)$ is a covariant representation of (G, η, X).

(2) Since (π_A, v, H', V'), where π_A is the underlying $*$-representation of π_X, is unitarily equivalent to the covariant Stinespring construction associated to φ, there is a unitary operator $U'_1 \in L(H_\Phi, H')$ such that $V' = U'_1 V_\Phi$, $v_t U'_1 = U'_1 v_t^\Phi$ for all $t \in G$, and $U'_1\pi_\varphi(a) = \pi_A(a)U'_1$ for all $a \in A$. On the other hand, $(\pi_\Phi, H_\Phi, K_\Phi, V_\Phi, W_\Phi)$ is the Stinespring construction associated to Φ, and then by Theorem 2.2 (2), there are two unitary operators $U_1 \in L(H_\Phi, H')$ and $U_2 \in L(K_\Phi, K')$ such that: $U_2\pi_\Phi\left(x\right) = \pi_X\left(x\right)U_1$ for all $x \in X$, $V' = U_1 V_\Phi$ and $W' = U_2 W_\Phi$. Moreover,

$$U_2\left(\sum_{i=1}^{n}\pi_\Phi\left(x_i\right)V_\Phi h_i\right) = \sum_{i=1}^{n}\pi_X\left(x_i\right)V' h_i$$

for all $x_1,...,x_n \in X$ and for all $h_1,...,h_n \in H$, whence

$$\begin{aligned} w_tU_2\left(\sum_{i=1}^{n}\pi_\Phi\left(x_i\right)V_\Phi h_i\right) &= w_t\left(\sum_{i=1}^{n}\pi_X\left(x_i\right)V'h_i\right)=\sum_{i=1}^{n}\pi_X\left(\eta_t\left(x_i\right)\right)v_tV'h_i \\ &= \sum_{i=1}^{n}\pi_X\left(\eta_t\left(x_i\right)\right)V'u_th_i = U_2\left(\sum_{i=1}^{n}\pi_\Phi\left(\eta_t\left(x_i\right)\right)V_\Phi u_th_i\right) \\ &= U_2\left(\sum_{i=1}^{n}w_t^\Phi\pi_\Phi\left(x_i\right)V_\Phi h_i\right)=U_2w_t^\Phi\left(\sum_{i=1}^{n}\pi_\Phi\left(x_i\right)V_\Phi h_i\right) \end{aligned}$$

and so $w_tU_2 = U_2w_t^\Phi$ for all $t \in G$.

From $U_2\pi_\Phi(x) = \pi_X(x)U_1$ for all $x \in X$, we deduce that $U_1\pi_\varphi(a) = \pi_A(a)U_1$ for all $a \in A$ and then

$$U_1'\left(\pi_\varphi(a)V_\Phi h\right) = \pi_A(a)U_1'V_\Phi h = \pi_A(a)V'h = \pi_A(a)U_1V_\Phi h = U_1\left(\pi_\varphi(a)V_\Phi h\right)$$

for all $a \in A$ and for all $h \in H$. From this relation, since $[\pi_\varphi(A)V_\Phi H] = H$, we deduce that $U_1 = U_1'$, and the assertion is proved. □

4. Covariant completely positive maps and crossed products of Hilbert C^*-modules

Let (G,η,X) be a dynamical system on Hilbert C^*-modules. The linear space $C(G,X)$ of all continuous functions from G to X with compact support has a structure of pre-Hilbert $G\times_{\alpha^\eta}A$-module with the action of $G\times_{\alpha^\eta}A$ on $C(G,X)$ given by

$$\left(\widehat{x}f\right)(s) = \int_G \widehat{x}\left(t\right)\alpha_t^\eta\left(f\left(t^{-1}s\right)\right)dt$$

for all $\widehat{x} \in C(G,X)$ and $f \in C(G,A)$ and the inner product given by

$$\left\langle \widehat{x},\widehat{y}\right\rangle(s) = \int_G \alpha_{t^{-1}}^\eta\left(\left\langle \widehat{x}(t),\widehat{y}\left(ts\right)\right\rangle\right)dt.$$

The crossed product of X by η, denoted by $G\times_\eta X$, is the Hilbert $G\times_{\alpha^\eta}A$-module obtained by the completion of the pre-Hilbert $G\times_{\alpha^\eta}A$-module $C(G,X)$ [4, 6]

Any covariant representation (π_X,v,w,H,K) of (G,η,X) induces a representation $(\pi_X\times v,H,K)$ of $G\times_\eta X$ such that

$$\left(\pi_X\times v\right)\left(\widehat{x}\right) = \int_G \pi_X\left(\widehat{x}\left(t\right)\right)v_t dt$$

for all $\widehat{x} \in C(G,X)$. Moreover, the underlying $*$-representation of $\pi_X \times v$ is the integral form of the covariant representation (π_A,v,H) of (G,α^η,A) induced by (π_X,v,H,K) [6].

Remark 4.1. *If (π_X,v,w,H,K) is a nondegenerate covariant representation of (G,η,X), then its integral form $(\pi_X\times v,H,K)$ is nondegenerate*

Indeed, let $f \in C(G,A)$ and $x \in X$. Then $f_x \in C(G,X)$, where $f_x(s) = xf(s)$,

$$\left(\pi_X\times v\right)\left(f_x\right) = \int_G \pi_X\left(xf\left(t\right)\right)v_t dt = \int_G \pi_X\left(x\right)\pi_A\left(f\left(t\right)\right)v_t dt = \pi_X\left(x\right)\left(\pi_A\times v\right)\left(f\right)$$

and

$$\left(\pi_X\times v\right)\left(f_x\right)^* = \left(\pi_A\times v\right)\left(f\right)^*\pi_X\left(x\right)^*.$$

From these facts and taking into account that $(\pi_A\times v,H,K)$ and (π_X,v,H,K) are nondegenerate we deduce that $\left[\left(\pi_X\times v\right)\left(X\right)H\right] = K$ and $\left[\left(\pi_X\times v\right)\left(X\right)^*K\right] = H$.

Proposition 4.2. *Let $\Phi : X \to L(H,K)$ be a completely positive map, (u',u)-covariant with respect to (G,η,X). Then there is a completely positive map $\widehat{\Phi} : G\times_{\eta} X \to L(H,K)$ such that*

$$\widehat{\Phi}\left(\widehat{x}\right) = \int_G \Phi\left(\widehat{x}\left(t\right)\right) u_t dt$$

for all $\widehat{x} \in C(G,X)$. Moreover, the completely positive map associated to $\widehat{\Phi}$ is the map $\widehat{\varphi} : G\times_{\alpha^{\eta}} A \to L(H)$ such that

$$\widehat{\varphi}\left(f\right) = \int_G \varphi\left(f\left(t\right)\right) u_t dt$$

for all $f \in C\left(G,A\right)$.

Proof. Let $\left(\pi_{\Phi}, v^{\Phi}, w^{\Phi}, H_{\Phi}, K_{\Phi}, V_{\Phi}, W_{\Phi}\right)$ be the covariant Stinespring construction associated to Φ. Consider the map $\widehat{\Phi} : G\times_{\eta} X \to L(H,K)$ defined by

$$\widehat{\Phi}\left(z\right) = W_{\Phi}^{*}\left(\pi_{\Phi}\times v^{\Phi}\right)\left(z\right) V_{\Phi}.$$

By Proposition 3.3 , $\widehat{\Phi}$ is completely positive and

$$\begin{aligned}\widehat{\Phi}\left(\widehat{x}\right) &= W_{\Phi}^{*}\left(\pi_{\Phi}\times v^{\Phi}\right)\left(\widehat{x}\right) V_{\Phi} = \int_G W_{\Phi}^{*}\pi_{\Phi}\left(\widehat{x}\left(t\right)\right) v_t^{\Phi} V_{\Phi} dt \\ &= \int_G W_{\Phi}^{*}\pi_{\Phi}\left(\widehat{x}\left(t\right)\right) V_{\Phi} u_t dt = \int_G \Phi\left(\widehat{x}\left(t\right)\right) u_t dt\end{aligned}$$

for all $\widehat{x} \in C(G,X)$. Since $(\pi_{\varphi}\times v^{\varphi}, H_{\Phi}, V_{\Phi})$ is the Stinespring construction associated to $\widehat{\varphi} : G\times_{\alpha^{\eta}} A \to L(H)$ with

$$\widehat{\varphi}\left(f\right) = \int_G \varphi\left(f\left(t\right)\right) u_t dt$$

for all $f \in C\left(G,A\right)$, we have

$$\begin{aligned}\left\langle \widehat{\Phi}\left(z_1\right), \widehat{\Phi}\left(z_1\right)\right\rangle &= V_{\Phi}^{*}\left\langle \left(\pi_{\Phi}\times v^{\Phi}\right)\left(z_1\right), \left(\pi_{\Phi}\times v^{\Phi}\right)\left(z_1\right)\right\rangle V_{\Phi} \\ &= V_{\Phi}^{*}\left(\pi_{\varphi}\times v^{\Phi}\right)\left(\left\langle z_1, z_1\right\rangle\right) V_{\Phi} = \widehat{\varphi}\left(\left\langle z_1, z_1\right\rangle\right)\end{aligned}$$

for all $z_1, z_2 \in G\times_{\eta} X$. Therefore, the completely positive map associated to $\widehat{\Phi}$ is $\widehat{\varphi}$. □

Remark 4.3. *Let $\Phi : X \to L(H,K)$ be a completely positive map, (u',u)-covariant with respect to (G,η,X). If $\left(\pi_{\Phi}, v^{\Phi}, w^{\Phi}, H_{\Phi}, K_{\Phi}, V_{\Phi}, W_{\Phi}\right)$ is the covariant Stinespring construction associated to Φ, then $\left(\pi_{\Phi}\times v^{\Phi}, H_{\Phi}, K_{\Phi}, V_{\Phi}, W_{\Phi}\right)$ is the Stinespring construction associated to $\widehat{\Phi}$. Indeed, we have*

$$\begin{aligned}&\left[\left\{\left(\pi_{\Phi}\times v^{\Phi}\right)\left(f_x\right) V_{\Phi} h; x \in X, f \in C(G,A), h \in H\right\}\right] \\ = \; &\left[\pi_{\Phi}\left(X\right)\pi_{\varphi}(C(G,A)) V_{\Phi} H\right] = \left[\pi_{\Phi}\left(x\right) H\right] = K\end{aligned}$$

and

$$\begin{aligned}&\left[\left\{\left(\pi_{\Phi}\times v^{\Phi}\right)\left(f_x\right)^{*} W_{\Phi}^{*} k; x \in X, f \in C(G,A), k \in K\right\}\right] \\ = \; &\left[\pi_{\varphi}(C(G,A))^{*}\pi_{\Phi}\left(X\right)^{*} W_{\Phi}^{*} K\right] = \left[\pi_{\varphi}(C(G,A)) H\right] = H.\end{aligned}$$

From these relations and taking into account that the map $\widehat{\Phi}$ is defined by $\widehat{\Phi}\left(z\right) = W_{\Phi}^{*}\left(\pi_{\Phi}\times v^{\Phi}\right)\left(z\right) V_{\Phi}$, we deduce that $\left(\pi_{\Phi}\times v^{\Phi}, H_{\Phi}, K_{\Phi}, V_{\Phi}, W_{\Phi}\right)$ is the Stinespring construction associated to $\widehat{\Phi}$ (see, Theorem 2.2).

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